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Position operators for the relativistic non-interacting n-particle system

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Abstract. The subject under consideration is a system of n non-interacting particles with spin, being described by a tensor product of n unitary irreducible massive representations of the Poincaré group. A unitary transformation that effects a separation of centre-of-mass variables and internal variables is constructed. By means of this transformation, for any of the n particles there are constructed two position operators both acting on the internal variables only. The first of these operators (called the internal position) commutes with all spatial translations and with the centre-of-mass position, the second one (called the impact position) commutes moreover with all time translations. It is argued that the impact position may be interpreted as the internal position taken at the instant when the n particles are most closely together. By considering spatial separation properties, it is shown that both the internal position and the impact position of any particle depend essentially on the location of all particles of the system.

1. Introduction

In this paper we investigate the kinematics of a system of n non-interacting relativistic massive particles with arbitrary spin in the framework of (mathematically rigorous) quantum theory. In more detail, we derive a simple standard form† for a tensor product of n irreducible unitary representations of the (quantum mechanical) Poincaré group and thereby we are naturally led to the definition of two types of position operators that have easily understandable non-relativistic classical counterparts.

Particularly, the 'internal centre-of-mass (CM) position operators' defined by Osborn (1968) for the two-body system are generalised for any finite number of particles resulting in operators $\boldsymbol{R}_j$, $j \in \{1, \ldots, n\}$, that shall be referred to as internal positions. The author's reason for being interested in these operators is that they can be used as dynamical variables in constructing Poincaré invariant model theories of directly interacting relativistic particles. The direct-interaction approach to relativistic particle dynamics that takes into account the dynamical degrees of freedom of particles and ignores those of fields, originated with papers of Dirac (1949), Bakamjian and Thomas (1953), and Foldy (1961); later on, various considerably differing versions of this approach were discussed by several authors. More recent papers on the subject are, for instance, Foldy and Krajcik (1975), Coester and Havas (1976),

† Steps towards this standard form were presented by Wightman (1960), in the proof of theorem 4.3, and by Chakrabarti (1964).

and Mutze (1978). In the present paper, however, no applications to direct-interaction theory shall be discussed.

Further, we introduce vector operators $\boldsymbol{Q}_j$ the non-relativistic classical counterparts of which describe the position of the jth particle relative to the CM taken at that instant at which the 'mean diameter' $2(\Sigma_j\mu_j\boldsymbol{r}_j^2)^{1/2}$ of the n-particle system (see (1.1) for the meaning of the symbols) attains its smallest possible value. Therefore, the Poincaré invariant operator $F\equiv 4\Sigma_j\mu_j\boldsymbol{Q}_j\cdot\boldsymbol{Q}_j$ is expected to represent the square of some kind of 'minimal mean diameter' of the system. More precisely, the Poincaré invariant condition $\langle\psi|F\psi\rangle\leqslant d^2\langle\psi|\psi\rangle$ is expected to be an appropriate mathematical formulation of the vague statement that ψ represents a state of n freely moving particles that, at some instant, are approximately localised all together in the interior of a sphere of diameter d. If long-range forces are absent, the asymptotic final states resulting from decays and collisions should be such approximately localised states with a microscopic value of d.

In order to motivate the subsequent relativistic constructions, we first discuss the non-relativistic classical counterparts $\boldsymbol{r}_j$ and $\boldsymbol{q}_j$, $j\in\{1,\ldots,n\}$, of the $\boldsymbol{R}_j$ and $\boldsymbol{Q}_j$ mentioned above. Let the position, momentum, and mass of the jth of n free classical non-relativistic (spinless) particles be denoted respectively by $\boldsymbol{x}_j$, $\boldsymbol{p}_j$, and m_j. The time dependence shall be displayed only if necessary. We introduce centre-of-mass and internal coordinates as follows:

$$\begin{aligned}
&\boldsymbol{x}\equiv\sum_j\mu_j\boldsymbol{x}_j \qquad \text{with } \mu_j\equiv\frac{m_j}{\Sigma_i m_i}, \qquad \boldsymbol{p}\equiv\sum_j\boldsymbol{p}_j,\\
&\boldsymbol{r}_j\equiv\boldsymbol{x}_j-\boldsymbol{x}, \qquad \boldsymbol{k}_j\equiv\boldsymbol{p}_j-\mu_j\boldsymbol{p}.
\end{aligned} \tag{1.1†}$$

Obviously we have

$$\sum_j\mu_j\boldsymbol{r}_j=0, \qquad \sum_j\boldsymbol{k}_j=0. \tag{1.2}$$

We note the Poisson bracket relations, which are trivial consequences of $\{x_i^\alpha,p_j^\beta\}=\delta_{\alpha\beta}\delta_{ij}$, $\{x_i^\alpha,x_j^\beta\}=\{p_i^\alpha,p_j^\beta\}=0$:

$$\{r_i^\alpha,k_j^\beta\}=\delta_{\alpha\beta}(\delta_{ij}-\mu_j), \qquad \{r_i^\alpha,r_j^\beta\}=\{k_i^\alpha,k_j^\beta\}=0, \tag{1.3}$$

$$\{x^\alpha,p^\beta\}=\delta_{\alpha\beta}, \qquad \{x^\alpha,x^\beta\}=\{p^\alpha,p^\beta\}=0, \tag{1.4}$$

$$\{r_i^\alpha,x^\beta\}=\{k_i^\alpha,x^\beta\}=\{r_i^\alpha,p^\beta\}=\{k_i^\alpha,p^\beta\}=0. \tag{1.5}$$

The energy e and the angular momentum $\boldsymbol{j}$ may be expressed in terms of internal variables and CM variables as follows:

$$e=\Big(\sum_j\boldsymbol{k}_j^2/2m_j\Big)+\boldsymbol{p}^2/2m \qquad \text{with } m\equiv\sum_j m_j, \tag{1.6}$$

$$\boldsymbol{j}=\Big(\sum_j\boldsymbol{r}_j\times\boldsymbol{k}_j\Big)+\boldsymbol{x}\times\boldsymbol{p}. \tag{1.7}$$

† Unspecified sums and products always run from 1 to n. Equations that contain free Latin and (or) Greek indices should always be understood as stated for all values of these indices contained in the sets $\{1,\ldots,n\}$ and $\{1,2,3\}$ respectively.

The time derivatives of the variables are given by

$$\dot{\boldsymbol{r}}_j = \boldsymbol{v}_j \qquad \text{with } \boldsymbol{v}_j \equiv \boldsymbol{k}_j/m_j, \quad \dot{\boldsymbol{k}}_j = 0 \tag{1.8}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{p}/m, \qquad \dot{\boldsymbol{p}} = 0. \tag{1.9}$$

Therefore, the quantity

$$t - \Big(\sum_j \boldsymbol{k}_j \cdot \boldsymbol{r}_j(t)\Big)\Big(\sum_j \boldsymbol{k}_j \cdot \boldsymbol{v}_j\Big)^{-1} \tag{1.10}$$

is a constant of motion. Let t_0 be the value of this quantity. Then it is easily shown that the quantity

$$d(t) \equiv 2\Big(\sum_j \mu_j \boldsymbol{r}_j(t)^2\Big)^{1/2}, \tag{1.11}$$

which is obviously a kind of diameter of the system, satisfies $d(t) > d(t_0)$ for all $t \neq t_0$, whenever the internal energy $e_{\text{in}} = \Sigma_j \boldsymbol{k}_j^2/2m_j$ is different from zero. Roughly speaking, t_0 is the instant at which the particles come most closely together. Therefore, let us call t_0 the instant of impact. It should be noted that this notion refers to the n-particle system as a whole: any subset of particles may reach its state of closest approach, or may even collide, at another instant than that of (n-particle) impact. The internal position $\boldsymbol{r}_j(t_0)$ of the jth particle 'at impact' shall be denoted by $\boldsymbol{q}_j$ and shall be called the impact position of this particle. This quantity is obviously a constant of motion, is invariant with respect to spatial translations and Galilean 'boosts', and is a vector with respect to rotations. Hence, the quantities $(\boldsymbol{q}_j - \boldsymbol{q}_l)^2$ are Galilean invariants. From $\boldsymbol{r}_j(t_0) = \boldsymbol{r}_j(t) + (t_0 - t)\boldsymbol{v}_j$ we obtain by (1.8) and (1.10)

$$\boldsymbol{q}_j \equiv \boldsymbol{r}_j(t_0) = \boldsymbol{r}_j - \boldsymbol{v}_j\Big(\sum_i \boldsymbol{k}_i \cdot \boldsymbol{r}_i\Big)\Big(\sum_i \boldsymbol{k}_i \cdot \boldsymbol{v}_i\Big)^{-1}. \tag{1.12}$$

Finally we note that the quantity

$$\tau(t) \equiv \Big(\sum_j \boldsymbol{k}_j \cdot \boldsymbol{r}_j(t)\Big)\Big(\sum_j \boldsymbol{k}_j \cdot \boldsymbol{v}_j\Big)^{-1} = t - t_0, \tag{1.13}$$

which is the internal virial (the virial $\Sigma_j \boldsymbol{p}_j \cdot \boldsymbol{x}_j$ may be written as a sum of a CM contribution $\boldsymbol{p} \cdot \boldsymbol{x}$ and an internal part $\Sigma_j \boldsymbol{k}_j \cdot \boldsymbol{r}_j$; t_0 is just the instant at which the latter part vanishes) divided by twice the internal energy, is canonically conjugate to the energy:

$$\{\tau, e\} = \{\tau, e_{\text{in}}\} = 1. \tag{1.14}$$

2. Mathematical description of the non-interacting relativistic n-particle system

In this preparatory section, we collect some facts on the unitary continuous representations (reps) of the Poincaré group $\mathscr{P}$ (cf the appendix). Any rep U of $\mathscr{P}$ determines the self-adjoint operators E (energy), $\boldsymbol{P}$ (momentum), $\boldsymbol{J}$ (angular momentum), and $\boldsymbol{N}$ ('boost') via Stone's theorem by the equations

$$\begin{aligned} &U(a, 1) = \exp(\mathrm{i}a^0 E - \mathrm{i}\boldsymbol{a} \cdot \boldsymbol{P}), \\ &U(0, \mathrm{e}^{\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{\sigma}}) = \exp(2\mathrm{i}\boldsymbol{u} \cdot \boldsymbol{J}), \qquad U(0, \mathrm{e}^{\boldsymbol{u}\cdot\boldsymbol{\sigma}}) = \exp(2\mathrm{i}\boldsymbol{u} \cdot \boldsymbol{N}). \end{aligned} \tag{2.1}$$

We shall be concerned only with reps of $\mathscr{P}$ for which the operators E and $M^2 \equiv E^2 - \boldsymbol{P}^2$ are positive and for which the spectrum of the mass operator M has a strictly positive lower bound. Such reps shall simply be called positive. Obviously, any tensor product of positive reps is positive. For positive reps, the operators $\boldsymbol{S}$ of spin and $\boldsymbol{X}$ of CM position are well defined and show the following property: there is a linear space $\mathscr{D}$ such that: (i) for each member A of the set $\{E, E^{-1}, M^{-1}, (M+E)^{-1}, P^\alpha, J^\alpha, N^\alpha, X^\alpha, S^\alpha : \alpha \in \{1, 2, 3\}\}$ of operators we have $\mathscr{D} \subset$ domain A, $A\mathscr{D} \subset \mathscr{D}$, A is essentially self-adjoint on $\mathscr{D}$; and (ii) we have the following equations on $\mathscr{D}$:

$$\begin{aligned} &\boldsymbol{X} = \boldsymbol{T} - \boldsymbol{P} \times (\boldsymbol{J} - \boldsymbol{T} \times \boldsymbol{P}) M^{-1}(M+E)^{-1} \qquad \text{with } \boldsymbol{T} \equiv \tfrac{1}{2}(E^{-1}\boldsymbol{N} + \boldsymbol{N}E^{-1}), \\ &\boldsymbol{N} = \tfrac{1}{2}(E\boldsymbol{X} + \boldsymbol{X}E) + \boldsymbol{P} \times (\boldsymbol{J} - \boldsymbol{X} \times \boldsymbol{P})(M+E)^{-1}, \\ &\boldsymbol{S} = \boldsymbol{J} - \boldsymbol{X} \times \boldsymbol{P}. \end{aligned} \tag{2.2}$$

Let an irreducible positive rep U of $\mathscr{P}$ with mass m (i.e. $M = m1, m > 0$) and spin s (i.e. $\boldsymbol{S}^2 = s(s+1)1, s \in \{0, \frac{1}{2}, 1, \dots\}$) be given on a Hilbert space H. Then there is a unitary transformation $V: H \to \mathrm{L}^2(\mathbb{R}^3, \mathbb{C}^{2s+1})$ (which, by Schur's lemma, is determined by U up to a phase factor) such that $VUV^{-1} = U_{m,s}$ with

$$\begin{aligned} &(U_{m,s}(a, A)\psi)(\boldsymbol{p}) \\ &\quad = \mathrm{e}^{\mathrm{i}p.a}((A^{-1}p)^0/p^0)^{1/2} D^{(s)}(\mathrm{R}(p, A))\psi(\mathrm{TVP}(A^{-1}p)) \qquad \text{for all } (a, A) \in \mathscr{P}, \end{aligned} \tag{2.3}$$

where p is the four-vector $(p^0, \boldsymbol{p}) = ((m^2 + \boldsymbol{p}^2)^{1/2}, \boldsymbol{p})$, $D^{(s)}$ is the unitary irreducible rep of SU(2) on $\mathbb{C}^{2s+1}$, and TVP denotes the mapping that assigns to a four-vector q its three-vector part $\boldsymbol{q}$; the Wigner rotation $\mathrm{R}(p, A)$ and the action of $A \in \mathrm{SL}(2, \mathbb{C})$ on four-vectors are explained in the appendix.

With a system of non-interacting distinguishable† particles with non-zero masses $m_1, \dots, m_n$ and spins $s_1, \dots, s_n$ there is associated a tensor product $U = \otimes_j U_j$ (cf the footnote to equation (1.1)) of irreducible positive reps, where U_j is (unitarily) equivalent to U_{m_j, s_j} (see (2.3)). Due to its tensor product structure, the rep U extends canonically to an irreducible rep $\bar{U}$ of the direct product group $\mathscr{P} \times \dots \times \mathscr{P}$ (n times): $\bar{U}(g_1, \dots, g_n) \equiv \otimes_j U_j(g_j)$ for all $(g_1, \dots, g_n) \in \mathscr{P} \times \dots \times \mathscr{P}$. The operators $E_j, \boldsymbol{P}_j, \boldsymbol{J}_j, \boldsymbol{N}_j, \boldsymbol{X}_j$, and $\boldsymbol{S}_j$ referring to the jth particle are defined as in (2.1) with U replaced by the positive rep $g \mapsto \bar{U}(1, \dots, 1, g, 1, \dots, 1)$ with g at the jth position. Let $H = \otimes_j H_j$ be the Hilbert space of U, then there is a unitary transformation $V: H \to \mathrm{L}^2(\mathbb{R}^{3n}, \otimes_j \mathbb{C}^{2s_j+1})$ (which is determined by $\bar{U}$ up to phase factor) such that $V\bar{U}V^{-1} \equiv \bar{U}'$ is given by

$$\begin{aligned} &(\bar{U}'(a_1, A_1; \dots; a_n, A_n)\psi)(\boldsymbol{p}_1, \dots, \boldsymbol{p}_n) \\ &\quad = \prod_j \mathrm{e}^{\mathrm{i}p_j.a_j}((A_j^{-1}p_j)^0/p_j^0)^{1/2} \bigotimes_j D^{(s_j)}(\mathrm{R}(p_j, A_j)) \\ &\quad \times \psi(\mathrm{TVP}(A_1^{-1}p_1), \dots, \mathrm{TVP}(A_n^{-1}p_n)). \end{aligned} \tag{2.4}$$

The essential tool for defining the operators we are concerned with in this paper is the well known 'transformation to the rest frame', which shall be described now. Defining

$$K^{(n)} \equiv \{(\boldsymbol{k}_1, \dots, \boldsymbol{k}_n) \in \mathbb{R}^{3n} : \boldsymbol{k}_1 + \dots + \boldsymbol{k}_n = 0\} \tag{2.5}$$

† Since we shall treat all particles on equal footing, the inclusion of identical particles shall be possible in a straightforward manner.

we introduce the mapping

$$\gamma\colon \mathbb{R}^{3n} \to K^{(n)} \times \mathbb{R}^3, \qquad \gamma(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \equiv (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}),$$

where $(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p})$ is given by the following chain of formulae:

$$\begin{aligned} &p_j^0 \equiv (m_j^2 + \boldsymbol{p}_j^2)^{1/2}, \qquad p_j \equiv (p_j^0, \boldsymbol{p}_j), \qquad p \equiv \sum_j p_j, \\ &k_j \equiv A(p)^{-1} p_j, \end{aligned} \tag{2.6}$$

where the matrix $A(p) \in \mathrm{SL}(2, \mathbb{C})$ is defined in the appendix. The mapping γ is a bijection and $\gamma^{-1}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ is given by

$$\begin{aligned} &k_j^0 \equiv (m_j^2 + \boldsymbol{k}_j^2)^{1/2}, \qquad m \equiv \sum_j k_j^0, \\ &p \equiv ((m^2 + \boldsymbol{p}^2)^{1/2}, \boldsymbol{p}), \qquad p_j \equiv A(p) k_j. \end{aligned} \tag{2.7}$$

The physical meaning of the k_j is obvious: let the four-momenta p_j be measured in a Lorentz frame F, then k_j is the four-momentum of the jth particle measured in a different Lorentz frame F' that is a rest frame in the sense that the total (three-) momentum of the particle system is found to be zero in F'. Since the manifold $K^{(n)} \times \mathbb{R}^3$ is a submanifold of $\mathbb{R}^{3n+3}$, it carries a natural Borel measure ζ given by

$$\mathrm{d}\zeta(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) = n^{3/2} \delta\Big(\sum_j \boldsymbol{k}_j\Big)\, \mathrm{d}\boldsymbol{k}_1 \ldots \mathrm{d}\boldsymbol{k}_n\, \mathrm{d}\boldsymbol{p}. \tag{2.8}$$

Denoting by λ the Lebesgue measure on $\mathbb{R}^{3n}$, we may compare the Borel measures $\gamma(\lambda)$ and ζ on $K^{(n)} \times \mathbb{R}^3$. These measures are equivalent, with a Radon-Nikodým derivative given by

$$(\mathrm{d}\gamma(\lambda)/\mathrm{d}\zeta)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) = n^{-3/2} m (p^0)^{-1} \prod_j p_j^0 / k_j^0, \tag{2.9}$$

where we have used the convention that the quantities m_j, p_j, k_j, m, p, are related by the equations (2.6) and (2.7). This very economical convention will be adopted throughout the remainder of this paper. As is easily shown, the following mapping is unitary:

$$W\colon \mathrm{L}^2(\mathbb{R}^{3n}, \otimes_j \mathbb{C}^{2s_j+1}, \lambda) \to \mathrm{L}^2(K^{(n)} \times \mathbb{R}^3, \otimes_j \mathbb{C}^{2s_j+1}, \zeta),$$

$$\begin{aligned} &(W\psi)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) \\ &\quad \equiv [(\mathrm{d}\gamma(\lambda)/\mathrm{d}\zeta)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p})]^{1/2} \otimes_j D^{(s_j)}(\mathrm{R}(p_j, A(p))^{-1}) \psi(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n). \end{aligned} \tag{2.10}$$

The rep $W\bar{U}' W^{-1}$ of $\mathscr{P} \times \ldots \times \mathscr{P}$ has a rather complicated form, which will not be written down. However, the restriction of $W\bar{U}' W^{-1}$ to the diagonal subgroup $\{(g_1, \ldots, g_n) \in \mathscr{P}^n\colon g_1 = g_2 = \ldots = g_n\}$ attains a rather transparent form. For the sake of simplicity, let us denote this rep of $\mathscr{P}$ by U, just as the n-particle rep that we started with. We have

$$\begin{aligned} &(U(a, A)\psi)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) \\ &\quad = \mathrm{e}^{\mathrm{i}p.a} ((A^{-1}p)^0 / p^0)^{1/2} \otimes_j D^{(s_j)}(\mathrm{R}(p, A)) \\ &\qquad \times \psi(\mathrm{R}(p, A)^{-1}\boldsymbol{k}_1, \ldots, \mathrm{R}(p, A)^{-1}\boldsymbol{k}_n, \mathrm{TVP}(A^{-1}p)). \end{aligned} \tag{2.11}$$

Due to this structure, the rep U is easily decomposed into a direct integral of irreducible reps (2.3). Thus the mapping W has essentially solved the task for $\mathscr{P}$ that is solved for the rotation group by the Clebsch–Gordon coefficients.

Finally, let us relate the definition of W with Mackey's theory of induced representations (cf Mackey 1952 and the books by Warner 1972 and by Kirillov 1976). The rep $U_{m,s}$ in (2.3) may be considered as being induced by the rep $V_{m,s}$ of $\mathscr{G} \equiv \{(a, A) \in \mathscr{P} : A \in \mathrm{SU}(2)\}$ given by $V_{m,s}(a, A) \equiv \exp(ima^0)D^{(s)}(A)$. Therefore, the rep $\bar{U} = \otimes_j U_{m_j,s_j}$ of $\mathscr{P} \times \ldots \times \mathscr{P}$ is induced by the rep $\bar{V} \equiv \otimes_j V_{m_j,s_j}$ of $\mathscr{G} \times \ldots \times \mathscr{G}$. The intended effect of W is to transform $\bar{U}$ such that its restriction to the diagonal subgroup is easily identified as being induced by a rep of $\mathscr{G}$. From (2.11), we see that the inducing rep may be chosen as the rep $\tilde{V}$ being given on $\mathrm{L}^2(K^{(n)}, \otimes_j \mathbb{C}^{2s_j+1})$ by $(\tilde{V}(a, A)\psi)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) = \exp(ia^0 \Sigma_j k_j^0) \otimes_j D^{(s_j)}(A)\psi(A^{-1}\boldsymbol{k}_1, \ldots, A^{-1}\boldsymbol{k}_n)$. The reps that are obtained by restricting an induced rep to a closed subgroup are analysed in Mackey (1952, theorem 12.1). Our transformation W is rather naturally inferred from Mackey's analysis if one notes that each coset $\mathscr{G}g$, $g \in \mathscr{P}$, contains just one pure Lorentz transformation $A(p)$ (cf the appendix) and, hence, determines uniquely the three-vector $\mathrm{TVP}(p/m)$ as a consequence of which the double coset space in Mackey's theorem may be identified with the space of SU(2)-orbits in $K^{(n)}$ (the action being given by $B(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) \equiv (B\boldsymbol{k}_1, \ldots, B\boldsymbol{k}_n)$).

3. Definition of internal observables, and their properties

The form (2.11) of U suggests the definition of some operators that 'do not act on $\boldsymbol{p}$' and, therefore, commute with the operators $\boldsymbol{P}$ and $\boldsymbol{X}$ (cf (2.1) and (2.2)), which can easily be seen to satisfy

$$(\exp(i\boldsymbol{a} \cdot \boldsymbol{P})\psi)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) = \exp(i\boldsymbol{a} \cdot \boldsymbol{p})\psi(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}), \tag{3.1}$$

$$(\exp(i\boldsymbol{a} \cdot \boldsymbol{X})\psi)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) = \psi(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p} - \boldsymbol{a}). \tag{3.2}$$

The definition of internal momenta $\boldsymbol{K}_1, \ldots, \boldsymbol{K}_n$ and internal spins $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n$ according to the following equations is conventional:

$$(\exp(i\boldsymbol{a} \cdot \boldsymbol{K}_j)\psi)(\boldsymbol{k}_j, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) \equiv \exp(i\boldsymbol{a} \cdot \boldsymbol{k}_j)\psi(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}), \tag{3.3}$$

$$(\exp(2i\boldsymbol{a} \cdot \boldsymbol{Z}_j)\psi)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) \equiv \otimes_l D^{(s_l)}(e^{i\delta_{lj}\boldsymbol{a} \cdot \boldsymbol{\sigma}})\psi(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}). \tag{3.4}$$

Considering (3.1) and (3.2) and remembering that the $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n$ are restricted by the condition $\Sigma_j \boldsymbol{k}_j = 0$, we easily guess how to define $\boldsymbol{R}_j$ such that the quantum mechanical analogue of (1.3), particularly $[R_j^\alpha, K_l^\beta] = i\delta_{\alpha\beta}(\delta_{jl} - \mu_l)$, is satisfied:

$$(\exp(i\boldsymbol{a} \cdot \boldsymbol{R}_j)\psi)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) \equiv \psi(\boldsymbol{k}_1 - d_{j1}\boldsymbol{a}, \ldots, \boldsymbol{k}_n - d_{jn}\boldsymbol{a}, \boldsymbol{p})$$

with

$$d_{jl} \equiv \delta_{jl} - \mu_l, \qquad \mu_l \equiv \frac{m_l}{\Sigma_i m_i}. \tag{3.5}$$

By the definitions (3.3)–(3.5), we have for any $\boldsymbol{a} \in \mathbb{R}^3$ three self-adjoint operators $\boldsymbol{a} \cdot \boldsymbol{R}_j$, $\boldsymbol{a} \cdot \boldsymbol{K}_j$, and $\boldsymbol{a} \cdot \boldsymbol{Z}_j$, the first and the second of them being clearly unbounded; they all are internal operators in the sense that they commute with $\boldsymbol{X}$ and $\boldsymbol{P}$ (i.e. with all operators $\boldsymbol{b} \cdot \boldsymbol{X}$, $\boldsymbol{b} \cdot \boldsymbol{P}$, $\boldsymbol{b} \in \mathbb{R}^3$). We shall call $\boldsymbol{R}_j$ the internal position of the jth particle. For notational convenience, we introduce abbreviations for some operators that are derived from momenta and energies:

$$\begin{aligned} &\boldsymbol{W} \equiv \boldsymbol{P}M^{-1}, && \boldsymbol{U} \equiv \boldsymbol{P}(M+E)^{-1}, \\ &\boldsymbol{V}_j \equiv \boldsymbol{K}_j(K_j^0)^{-1}, && K_j^0 \equiv (m_j^2+\boldsymbol{K}_j^2)^{1/2}, \\ &\boldsymbol{W}_j \equiv \boldsymbol{V}_j - \sum_l \mu_l \boldsymbol{V}_l, && \epsilon_j \equiv E_jE^{-1}. \end{aligned} \tag{3.6}$$

To get rid of domain problems in writing down algebraic relations between unbounded operators, we search for a subspace $\mathscr{D} \subset \mathrm{L}^2(K^{(n)} \times \mathbb{R}^3, \otimes_j \mathbb{C}^{2s_j+1}, \zeta)$ such that we have for all operators introduced up to now:

(*a*) $\mathscr{D} \subset$ domain A;
(*b*) $A\mathscr{D} \subset \mathscr{D}$;
(*c*) A is essentially self-adjoint on $\mathscr{D}$.

We choose for $\mathscr{D}$ the space of all infinitely differentiable $\otimes_j \mathbb{C}^{2s_j+1}$-valued functions on $K^{(n)} \times \mathbb{R}^3$ with compact support ($K^{(n)} \times \mathbb{R}^3$ should always be considered as a differentiable manifold with the global chart $\phi: K^{(n)} \times \mathbb{R}^3 \to \mathbb{R}^{3n}$, $\phi(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) = (\boldsymbol{k}_2, \ldots, \boldsymbol{k}_n, \boldsymbol{p})$). Then, the properties (*a*) and (*b*) are rather obviously satisfied, and (*c*) is easily proved by theorem VIII.11 of Reed and Simon (1972) since the unitary group is explicitly given for all operators under consideration. On the space $\mathscr{D}$, the definitions (3.3)–(3.5) imply

$$\begin{aligned} (\boldsymbol{a} \cdot \boldsymbol{K}_j\psi)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) &= \boldsymbol{a} \cdot \boldsymbol{k}_j\psi(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}), \\ (\boldsymbol{a} \cdot \boldsymbol{R}_j\psi)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\lambda}\psi(\boldsymbol{k}_1 - \lambda d_{j1}\boldsymbol{a}, \ldots, \boldsymbol{k}_n - \lambda d_{jn}\boldsymbol{a}, \boldsymbol{p})|_{\lambda=0}, \\ (\boldsymbol{a} \cdot \boldsymbol{Z}_j\psi)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p}) &= (2\mathrm{i})^{-1}\frac{\mathrm{d}}{\mathrm{d}\lambda} \otimes_l D^{(s_l)}(\mathrm{e}^{\lambda \mathrm{i}\delta_{lj}\boldsymbol{a}\cdot\boldsymbol{\sigma}})\psi(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{p})|_{\lambda=0}. \end{aligned} \tag{3.7}$$

By straightforward calculation we verify on $\mathscr{D}$

$$\sum_j \mu_j \boldsymbol{R}_j = 0, \qquad \sum_j \boldsymbol{K}_j = 0, \tag{3.8}$$

$$[R_j^\alpha, K_l^\beta] = \mathrm{i}\delta_{\alpha\beta}(\delta_{jl} - \mu_l), \qquad [R_j^\alpha, R_l^\beta] = [K_j^\alpha, K_l^\beta] = 0, \tag{3.9}$$

$$\boldsymbol{J} = \boldsymbol{X} \times \boldsymbol{P} + \sum_j (\boldsymbol{Z}_j + \boldsymbol{R}_j \times \boldsymbol{K}_j). \tag{3.10}$$

These equations are the strict quantum mechanical analogues of the classical equations (1.2), (1.3), and (1.7) (note that only spin-zero particles are considered in § 1).

Further, we obtain the following laws of motion (with $U(t) \equiv U(t, \boldsymbol{0}, 1)$), which are counterparts of (1.8) and (1.9):

$$U(t)\boldsymbol{a} \cdot \boldsymbol{K}_jU(-t) = \boldsymbol{a} \cdot \boldsymbol{K}_j, \qquad U(t)\boldsymbol{a} \cdot \boldsymbol{Z}_jU(-t) = \boldsymbol{a} \cdot \boldsymbol{Z}_j, \tag{3.11}$$

$$U(t)\boldsymbol{a} \cdot \boldsymbol{R}_jU(-t) = \boldsymbol{a} \cdot \boldsymbol{R}_j + tME^{-1}\boldsymbol{a} \cdot \boldsymbol{W}_j, \tag{3.12}$$

$$U(t)\boldsymbol{a} \cdot \boldsymbol{P}U(-t) = \boldsymbol{a} \cdot \boldsymbol{P}, \qquad U(t)\boldsymbol{a} \cdot \boldsymbol{X}U(-t) = \boldsymbol{a} \cdot \boldsymbol{X} + t\boldsymbol{a} \cdot \boldsymbol{P}E^{-1}. \tag{3.13}$$

The Euclidean transformation properties of $\boldsymbol{K}_j$, $\boldsymbol{R}_j$, and $\boldsymbol{Z}_j$ are those of translationally invariant vector operators, i.e. we have for all $g=(0, \boldsymbol{b}, B)$, $\boldsymbol{b}\in\mathbb{R}^3$, $B\in SU(2)$ the equations

$$U(g)\boldsymbol{a}\,.\,\boldsymbol{A}U(g)^{-1}=\boldsymbol{a}'\,.\,\boldsymbol{A} \qquad \text{with } \boldsymbol{a}'\equiv B\boldsymbol{a}, \tag{3.14}$$

where $\boldsymbol{A}$ denotes any of the operators $\boldsymbol{K}_j$, $\boldsymbol{R}_j$, and $\boldsymbol{Z}_j$.

Now we are interested in the connection between the internal operators $\boldsymbol{K}_j$, $\boldsymbol{R}_j$, $\boldsymbol{Z}_j$, and the individual particle operators $\boldsymbol{P}_j$, $\boldsymbol{X}_j$, $\boldsymbol{S}_j$. Since the realisations (2.4) and (2.11) of U that give particular simple forms to these operators are connected by the explicitly known transformation (2.10), there is no fundamental difficulty in calculating the internal operators in terms of the individual ones. However, the transformation W is rather complicated so that the differentations that are contained in the definition of $\boldsymbol{R}_j$ and $\boldsymbol{Z}_j$ produce an avalanche of terms in the calculation. As a result we find that the following equations hold on $\mathscr{D}$:

$$\boldsymbol{K}_j=\boldsymbol{P}_j-\boldsymbol{W}(E_j-\boldsymbol{U}\,.\,\boldsymbol{P}_j). \tag{3.15}$$

$$\boldsymbol{R}_j=\boldsymbol{Y}_j-\sum_l \mu_l\boldsymbol{Y}_l$$

with

$$\boldsymbol{Y}_l\equiv\boldsymbol{X}_l+\tfrac{1}{2}\Big[\Big(X_l-\sum_i \boldsymbol{X}_i\epsilon_i\Big)\,.\,\boldsymbol{W}\boldsymbol{V}_l+\boldsymbol{V}_1\boldsymbol{W}\,.\,\Big(\boldsymbol{X}_l-\sum_i\epsilon_i\boldsymbol{X}_i\Big)\Big]+\tfrac{1}{2}(\boldsymbol{X}_l\,.\,\boldsymbol{W}\boldsymbol{U}+\boldsymbol{U}\boldsymbol{W}\,.\,\boldsymbol{X}_l)$$
$$+a_l^1\boldsymbol{U}+a_l^2\boldsymbol{V}_l+a_l^3\boldsymbol{S}_l\times\boldsymbol{U}+a_l^4\boldsymbol{P}_l\times\boldsymbol{U}$$

with

$$a_l^1\equiv\alpha_l\boldsymbol{S}_l\,.\,(\boldsymbol{P}_l\times\boldsymbol{W}) \qquad \text{with } \alpha_l\equiv[(m_l+E_l)(m_l+K_l^0)]^{-1}, \tag{3.16}$$

$$a_l^2\equiv a_l^1+\sum_i[E(m_i+E_i)]^{-1}\boldsymbol{S}_i\,.\,(\boldsymbol{P}_i\times\boldsymbol{W}),$$

$$a_l^3\equiv\alpha_l\beta_l \qquad \text{with } \beta_l\equiv m_l+E_l+K_l^0+m_lM^{-1}E,$$

$$a_l^4\equiv\alpha_l\boldsymbol{S}_l\,.\,\boldsymbol{W}.$$

$$\boldsymbol{Z}_j=\boldsymbol{S}_j-\alpha_j[(\boldsymbol{P}_j\times\boldsymbol{U})\,.\,(\boldsymbol{P}_j\times\boldsymbol{W})\boldsymbol{S}_j-(\boldsymbol{P}_j\times\boldsymbol{U})\,.\,\boldsymbol{S}_j(\boldsymbol{P}_j\times\boldsymbol{W})+\beta_j(\boldsymbol{P}_j\times\boldsymbol{U})\times\boldsymbol{S}_j]. \tag{3.17}$$

For the two-particle system, a relative position operator was constructed by Osborn (1968). Comparing (3.16) with equation (3.33) of Osborn, we find that our operator $\boldsymbol{R}_1-\boldsymbol{R}_2$ coincides exactly with Osborn's operator $\boldsymbol{r}$. Moreover, Osborn's method can readily be generalised for $n>2$ to yield our operators $\boldsymbol{R}_j$. Therefore, we consider the following rep D of the (multiplicative) group of positive numbers

$$(D(r)\psi)(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n,\boldsymbol{p})=r^{3/2}\psi(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n,r\boldsymbol{p}), \tag{3.18}$$

which obviously satisfies $D(r)\boldsymbol{P}D(r)^{-1}=r\boldsymbol{P}$ and $D(r)\boldsymbol{R}_jD(r)^{-1}=\boldsymbol{R}_j$. Now it can be shown that we have for all $\psi\in\mathscr{D}$

$$D(r)\Big(\boldsymbol{X}_j-\sum_l\mu_l\boldsymbol{X}_l\Big)D(r)^{-1}\psi\to\boldsymbol{R}_j\psi, \qquad \text{as } r\to 0. \tag{3.19}$$

Since $D(e^{-\alpha})=\exp[\alpha i(\boldsymbol{X}\,.\,\boldsymbol{P}+\boldsymbol{P}\,.\,\boldsymbol{X})/2]$, this is just the limit procedure used by Osborn to obtain from an operator that commutes with $\boldsymbol{P}$ a new one that commutes with $\boldsymbol{P}$ and $\boldsymbol{X}$ and approximates the first one if applied to states with a small expectation value of $\boldsymbol{P}$. From a mathematical point of view, Osborn's treatment is obscured by an erroneous statement concerning the existence of an isometric limit of $D(e^{-\alpha})$ as $\alpha \to \infty$; as a matter of fact, this limit does not exist in the strong operator topology and is the zero operator in the weak operator topology.

4. Definition of impact positions

In the preceding section we have constructed a relativistic quantum mechanical analogue $\boldsymbol{R}_j$ of the classical internal position $\boldsymbol{r}_j$ introduced in § 1. In the present section we shall construct the analogue $\boldsymbol{Q}_j$ of the classical quantity $\boldsymbol{q}_j$ that was introduced in § 1 as the internal position of the jth particle at the instant of maximal particle concentration. As in the preceding section, this construction will actually not proceed by analogy but by guess, based on the suggestive form (2.11) of the rep U. Let us recall that the guiding principle in defining $\boldsymbol{R}_j$ was to fulfil prescribed commutation relations with $\boldsymbol{K}_j$; now we search for an operator $\boldsymbol{Q}_j$ that is as closely related to $\boldsymbol{R}_j$ as possible but commutes not only with $\boldsymbol{P}$ and $\boldsymbol{X}$ (as $\boldsymbol{R}_j$) but also with M and hence with E. Thus, $\boldsymbol{Q}_j$ shall be a constant of motion as is the classical q_j.

Since $\boldsymbol{a}\,.\,\boldsymbol{R}_j$ is a derivative along the constant vector field $r_{\boldsymbol{a},j}$ with

$$r_{\boldsymbol{a},j}(k) \equiv (\boldsymbol{a}d_{j1}, \ldots, \boldsymbol{a}d_{jn}) \qquad \text{for all } k \in K^{(n)}, \tag{4.1}$$

it is a rather natural idea to deform this vector field such that it becomes tangent to each of the surfaces

$$S_m \equiv \left\{ k \in K^{(n)}: \sum_j (m_j^2 + \boldsymbol{k}_j^2)^{1/2} = m \right\}, \; m > \sum_j m_j. \tag{4.2}$$

The derivative associated with this new vector field then will clearly commute with M. Obviously, there are many ways of carrying out such a deformation, but that to be described subsequently seems to be the most simple one. First we note that, the manifold $K^{(n)}$ being a linear space, the tangent space $T(k)$ at $k \in K^{(n)}$ coincides with $K^{(n)}$. For the tangent vector $u \equiv (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) \in T(k)$ to be tangent to S_m (with m chosen such that $k \in S_m$), we easily find the necessary and sufficient condition

$$\sum_j \boldsymbol{u}_j\,.\,\boldsymbol{k}_j / k_j^0 = 0. \tag{4.3}$$

This suggests we consider $T(k)$ as an inner product space (hence $K^{(n)}$ as a Riemann manifold) with the following k-dependent inner product:

$$\langle u|v\rangle \equiv \sum_j \boldsymbol{u}_j\,.\,\boldsymbol{v}_j / k_j^0 \qquad \text{for all } u, v \in T(k). \tag{4.4}$$

Then the tangent space of S_m at $k \in S_m$ is simply $\{u \in T(k): \langle u|k\rangle = 0\}$. Thus, as S_m was a sphere, the tangent space of S_m at k is orthogonal to the radius vector k. Therefore, it is geometrically appealing to project $r_{\boldsymbol{a},j}$ on the sphere-like surface S_m along the direction of k:

$$q_{\boldsymbol{a},j}(k) \equiv r_{\boldsymbol{a},j}(k) - k\langle r_{\boldsymbol{a},j}(k)|k\rangle / \langle k|k\rangle. \tag{4.5}$$

In fact we have $\langle q_{\boldsymbol{a},j}(k)|k\rangle = 0$, hence $q_{\boldsymbol{a},j}(k)$ is tangent to S_m for $k \in S_m$. Therefore, q may be considered as a smooth vector field on any of the submanifolds S_m; since these are compact manifolds, q is complete on each S_m (see Abraham 1967, theorem 7.14).

Since any $k \in K^{(n)}\backslash\{0\}$ belongs to just one S_m, q is complete on $K^{(n)}\backslash\{0\}$ and by Abraham (1967, p 41), there is a one-parameter group $\phi_{a,j}$ of diffeomorphisms of $K^{(n)}\backslash\{0\}$ such that

$$\mathrm{d}\phi_{a,j}(\lambda)(k)/\mathrm{d}\lambda\,|_{\lambda=0} = q_{a,j}(k) \qquad \text{for all } k \in K^{(n)}\backslash\{0\}. \tag{4.6}$$

Now we define a one-parameter group of unitary operators on $\mathrm{L}^2(K^{(n)} \times \mathbb{R}^3, \otimes_j \mathbb{C}^{s_j+1}, \zeta)$ by

$$(\exp(\mathrm{i}\lambda \boldsymbol{a} \cdot \boldsymbol{Q}_j)\psi)(k, \boldsymbol{p}) \equiv \alpha_{a,j}(\lambda, k)\psi(\phi_{a,j}(-\lambda)k, \boldsymbol{p}), \tag{4.7}$$

where the numerical factor α is determined by a Radon-Nikodým derivative: $\alpha_{a,j}(\lambda, k) \equiv [(\mathrm{d}\phi_{a,j}(-\lambda)\zeta'/\mathrm{d}\zeta')(k)]^{1/2}$ with $\mathrm{d}\zeta'(k) \equiv n^{3/2}\delta(\Sigma_j k_j)\,\mathrm{d}k_1 \ldots \mathrm{d}k_n$. Since this factor becomes singular at $k = 0$, it is clear that not every $\psi \in \mathscr{D}$ belongs to the domain of the operator $\boldsymbol{a} \cdot \boldsymbol{Q}_j$. As a substitute for $\mathscr{D}$ we define

$$\mathscr{D}_0 \equiv \{\psi \in \mathscr{D}\colon \exists \epsilon > 0(\forall k \in K^{(n)}, \boldsymbol{p} \in \mathbb{R}^3(\psi(k, \boldsymbol{p}) = 0 \text{ whenever}$$
$$\Sigma_j\, [(m_j^2 + \boldsymbol{k}_j^2)^{1/2} - m_j] > \epsilon))\}. \tag{4.8}$$

This space is obviously invariant under $\boldsymbol{a} \cdot \boldsymbol{Q}_j$ and $\exp(\mathrm{i}\boldsymbol{a} \cdot \boldsymbol{Q}_j)$ (and $\boldsymbol{a} \cdot \boldsymbol{R}_j$, but not under $\exp(\mathrm{i}\boldsymbol{a} \cdot \boldsymbol{R}_j)$). Hence $\boldsymbol{a} \cdot \boldsymbol{Q}_j$ is essentially self-adjoint on $\mathscr{D}_0$. In the remainder of this paper, any operator equation should be understood as an equation on $\mathscr{D}_0$ if it contains $\boldsymbol{Q}_j$, and as an equation on $\mathscr{D}$ otherwise. For $\psi \in \mathscr{D}_0$ we obviously have

$$\mathrm{i}(\boldsymbol{a} \cdot \boldsymbol{Q}_j\psi)(k, \boldsymbol{p}) = \mathrm{d}\alpha_{a,j}(\lambda, k)/\mathrm{d}\lambda\,|_{\lambda=0}\psi(k, \boldsymbol{p}) + \mathrm{d}\psi(k - \lambda q_{a,j}(k), \boldsymbol{p})/\mathrm{d}\lambda\,|_{\lambda=0};$$

since the first term on the right-hand side gives a skew-Hermitian contribution to $\boldsymbol{Q}_j$, the Hermitian part of the second term equals the left-hand side. Performing the λ-differentation we obtain (see (3.6) for notation)

$$\boldsymbol{Q}_j = \boldsymbol{R}_j - \tfrac{1}{2}\left[\boldsymbol{W}_j\Big(\sum_l \boldsymbol{K}_l \cdot \boldsymbol{V}_l\Big)^{-1} \sum_l \boldsymbol{K}_l \cdot \boldsymbol{R}_l + \mathrm{HC}\right]. \tag{4.9}$$

For convenience, we define the non-Hermitian operator

$$\boldsymbol{Q}'_j \equiv \boldsymbol{R}_j - \boldsymbol{W}_j\Big(\sum_l \boldsymbol{K}_l \cdot \boldsymbol{V}_l\Big)^{-1} \sum_l \boldsymbol{K}_l \cdot \boldsymbol{R}_l. \tag{4.10}$$

Thus $\boldsymbol{Q}_j$ is expressed in terms of $\boldsymbol{R}_l$, $\boldsymbol{K}_l$, $\boldsymbol{V}_l$ in just the same way as the classical non-relativistic $\boldsymbol{q}_j$ is expressed in terms of $\boldsymbol{r}_l$, $\boldsymbol{k}_l$, and $\boldsymbol{v}_l$ (note $\Sigma_l\, \mu_l \boldsymbol{v}_l = 0$, so that the classical analogue of $\boldsymbol{W}_j$ is $\boldsymbol{v}_j$)†. For $n = 2$, we obtain the simple expression

$$\boldsymbol{Q}'_1 - \boldsymbol{Q}'_2 = \boldsymbol{R}_1 - \boldsymbol{R}_2 - \boldsymbol{e}[\boldsymbol{e} \cdot (\boldsymbol{R}_1 - \boldsymbol{R}_2)] \qquad \text{with } \boldsymbol{e} := \boldsymbol{K}_1/|\boldsymbol{K}_1|. \tag{4.11}$$

This is just the equation for the minimal distance of two trajectories, one passing through $\boldsymbol{R}_1$ with direction $\boldsymbol{e}$, the other passing through $\boldsymbol{R}_2$ with direction $-\boldsymbol{e}$, i.e. the equation for the impact parameter of two particles moving with any (constant) velocity along these trajectories. By (4.9) and (3.9), we easily find

$$[Q_j^\alpha, K_l^\beta] = \mathrm{i}\left[\delta_{\alpha\beta} d_{jl} - W_j^\alpha K_l^\beta\Big(\sum_i \boldsymbol{K}_i \cdot \boldsymbol{V}_i\Big)^{-1}\right]. \tag{4.12}$$

Contrary to the corresponding situation for the $\boldsymbol{R}_j$, the commutator $[Q_j^\alpha, Q_l^\beta]$ does not

† The author's actual method of proceeding was to introduce the operators $\boldsymbol{R}_j$ and $\boldsymbol{Q}_j$ first and then to consider the classical quantities $\boldsymbol{r}_j$ and $\boldsymbol{q}_j$ in order to understand the quantum mechanical ones.

vanish even for $j = l$, as is rather evident since the $\boldsymbol{Q}_j$ are derivatives along non-constant vector fields. Although this commutator can be calculated straightforwardly, the result is rather complicated and shall not be written down. Correspondingly to (3.8) and (3.10), we have

$$\sum_j \mu_j \boldsymbol{Q}_j = 0, \qquad \boldsymbol{J} = \boldsymbol{X} \times \boldsymbol{P} + \sum_j (\boldsymbol{Z}_j + \boldsymbol{Q}_j \times \boldsymbol{K}_j). \tag{4.13}$$

The invariance properties of the $\boldsymbol{Q}_j$ are the following. For any $\boldsymbol{a} \in \mathbb{R}^3$ and $j \in \{1, \ldots, n\}$ the self-adjoint operator $\boldsymbol{a} \cdot \boldsymbol{Q}_j$ commutes (in the strict operator sense) with the mass operator, with $\boldsymbol{X}$, and with $\boldsymbol{P}$, hence with all translation operators $U(a, 1)$, $a \in \mathbb{R}^4$. With respect to rotations, equation (3.14) is valid with $\boldsymbol{Q}_j$ instead of $\boldsymbol{A}$. Now the rotationally invariant operators $\boldsymbol{Q}_j \cdot \boldsymbol{Q}_l + \mathrm{HC}$, $\boldsymbol{K}_j \cdot \boldsymbol{Q}_l + \mathrm{HC}$, and $\boldsymbol{Z}_j \cdot \boldsymbol{Q}_l$ can be defined on $\mathscr{D}_0$ and are easily shown to commute formally (i.e. on $\mathscr{D}_0$) with all generators of U, moreover then can be shown to commute on $\mathscr{D}_0$ with $U(g)$ for all $g \in \mathscr{P}$. Probably, the closures of these operators are self-adjoint operators that commute with $U(g)$ for all $g \in \mathscr{P}$. However, this is not yet proved.

Finally, a slight transcription of (4.10) may be instructive:

$$\boldsymbol{Q}'_j = \boldsymbol{R}_j - \mathrm{i}[M, \boldsymbol{R}_j]T \qquad \text{with } T \equiv \sum_l \boldsymbol{K}_l \cdot \boldsymbol{R}_l \Big(\sum_l \boldsymbol{K}_l \cdot \boldsymbol{V}_l\Big)^{-1}, \tag{4.14}$$

where the equation $[T, M] = \mathrm{i}$ should be noted. The natural question, whether T, as in the classical case, may be interpreted as an observable describing the (proper) time of maximal particle concentration, shall not be discussed here.

5. Spatial separation properties of the $\boldsymbol{R}_j$ and $\boldsymbol{Q}_j$

In this section we shall investigate how the unitary operator $\exp(\mathrm{i}\boldsymbol{b} \cdot \boldsymbol{P}_l)$ (that effects a displacement of the lth particle by a vector $\boldsymbol{b}$) transforms the position operators $\boldsymbol{R}_j$ and $\boldsymbol{Q}_j$. Thus we shall reveal a strange property of these operators that may be roughly expressed as follows. Both the internal position and the impact position of the jth particle depend essentially on the position of any of the other particles, even if this particle is arbitrarily far away. The reasons for this lack of spatial separability are similar to those for the corresponding defect of barycentric wave operators (i.e. wave operators ($\equiv$Møller operators) that commute, like $\boldsymbol{Q}_j$ and $\boldsymbol{R}_j$, with the operators $\boldsymbol{X}$ and $\boldsymbol{P}$ of the free-particle rep associated with the asymptotic states) that was pointed out by Mutze (1978).

From the last equations of (2.7) we easily infer

$$\boldsymbol{P}_l = \boldsymbol{K}_l + (K_l^0 + \boldsymbol{U} \cdot \boldsymbol{K}_l)\boldsymbol{W}. \tag{5.1}$$

Then, we obtain from the second equation of (3.7) and from (3.9) by a straightforward calculation

$$[R_j^\alpha, P_l^\beta] = \mathrm{i} d_{jl}[\delta_{\alpha\beta} + (V_l^\alpha + U^\alpha)W^\beta] - \mathrm{i} W_j^\alpha W^\beta \epsilon_l. \tag{5.2}$$

Since the right-hand side of this equation commutes with $\boldsymbol{P}_l$, we expect

$$\exp(-\mathrm{i}\boldsymbol{b} \cdot \boldsymbol{P}_l)\boldsymbol{a} \cdot \boldsymbol{R}_j \exp(\mathrm{i}\boldsymbol{b} \cdot \boldsymbol{P}_l) = \boldsymbol{a} \cdot \boldsymbol{R}_j - \mathrm{i}[\boldsymbol{b} \cdot \boldsymbol{P}_l, \boldsymbol{a} \cdot \boldsymbol{R}_j]. \tag{5.3}$$

In fact, this equation can be proved. Therefore, we have

$$\exp(-\mathrm{i}\boldsymbol{b} \cdot \boldsymbol{P}_l)\boldsymbol{a} \cdot \boldsymbol{R}_j \exp(\mathrm{i}\boldsymbol{b} \cdot \boldsymbol{P}_l) = \boldsymbol{a} \cdot \boldsymbol{R}_j - d_{jl}\boldsymbol{a} \cdot \boldsymbol{B}_l(\boldsymbol{b}) + \boldsymbol{a} \cdot \boldsymbol{W}_j \boldsymbol{b} \cdot \boldsymbol{W} \epsilon_l \tag{5.4}$$

with

$$\boldsymbol{B}_l(\boldsymbol{b}) \equiv \boldsymbol{b} + (\boldsymbol{V}_l + \boldsymbol{U})\boldsymbol{b} \,.\, \boldsymbol{W}.$$

Together with (4.9), this equation yields

$$\exp(-\mathrm{i}\boldsymbol{b} \,.\, \boldsymbol{P}_l)\boldsymbol{a} \,.\, \boldsymbol{Q}_j \exp(\mathrm{i}\boldsymbol{b} \,.\, \boldsymbol{P}_l) = \boldsymbol{a} \,.\, \boldsymbol{Q}_j - d_{jl}\boldsymbol{a} \,.\, \boldsymbol{B}_l(\boldsymbol{b}) - \boldsymbol{a} \,.\, \boldsymbol{W}_j\Big(\sum_i \boldsymbol{K}_i \,.\, \boldsymbol{V}_i\Big)^{-1} \boldsymbol{K}_l \,.\, \boldsymbol{B}_l(\boldsymbol{b}). \tag{5.5}$$

The strange feature of the $\boldsymbol{R}_j$ announced at the beginning of this section becomes evident from the following equation holding for $l \neq j$, $l \neq k$:

$$\exp(-\mathrm{i}\lambda\boldsymbol{b} \,.\, \boldsymbol{P}_l)\boldsymbol{a} \,.\, (\boldsymbol{R}_j - \boldsymbol{R}_k) \exp(\mathrm{i}\lambda\boldsymbol{b} \,.\, \boldsymbol{P}_l)\psi = \boldsymbol{a} \,.\, (\boldsymbol{R}_j - \boldsymbol{R}_k)\psi + \lambda\boldsymbol{a} \,.\, (\boldsymbol{V}_j - \boldsymbol{V}_k)\boldsymbol{b} \,.\, \boldsymbol{W}\epsilon_l\psi. \tag{5.6}$$

Although the second term on the right-hand side can be made arbitrarily small for any fixed value of λ by choosing a ψ with a sufficiently small expection value of $\boldsymbol{P}^2$, this term will always dominate (even diverge) if λ tends to infinity. Thus, removing one particle to infinity affects discontinuously the internal distance between any pair of the remaining particles. A similar conclusion is valid for the distance $\boldsymbol{Q}_i - \boldsymbol{Q}_j$. In this case, however, that conclusion is not astonishing since, by removing one particle to infinity, the instant of impact tends to infinity too.

6. Concluding remarks

Here let us formulate some conjectures that arose from the preceding investigations. Therefore, we consider the von Neumann algebras A_1, A_2, and A_3 of those bounded operators that commute respectively with $\{\boldsymbol{P}, \boldsymbol{X}\}$, $\{\boldsymbol{P}, \boldsymbol{X}, M\}$, and $\{\boldsymbol{P}, \boldsymbol{X}, M, \boldsymbol{J}\}$. The members of A_1, A_2, and A_3 correspond respectively to internal observables (i.e. to observables that are not sensitive to the CM motion of the system), to time-independent internal observables, and to Poincaré invariant observables. Now the announced conjectures say that the algebra A_i, $i \in \{1, 2, 3\}$, is generated by the following set G_i of self-adjoint operators:

$$G_1 = \{K_j^\alpha, R_j^\alpha, Z_j^\alpha : j \in \{1, \ldots, n\}, \alpha \in \{1, 2, 3\}\},$$

$$G_2 = \{K_j^\alpha, Q_j^\alpha, Z_j^\alpha : j \in \{1, \ldots, n\}, \alpha \in \{1, 2, 3\}\},$$

$$G_3 = \{\boldsymbol{Q}_j \,.\, \boldsymbol{Q}_l + \mathrm{HC}, \boldsymbol{Q}_j \,.\, \boldsymbol{K}_l + \mathrm{HC}, \boldsymbol{Q}_j \,.\, \boldsymbol{Z}_l, \boldsymbol{K}_j \,.\, \boldsymbol{K}_l, \boldsymbol{K}_j \,.\, \boldsymbol{Z}_l, \boldsymbol{Z}_j \,.\, \boldsymbol{Z}_l : j, l \in \{1, \ldots, n\}\}.$$

Finally, we recall that the operator $\boldsymbol{a} \,.\, \boldsymbol{Q}_j$ was constructed from a vector field on $K^{(n)}$ being tangent to the submanifolds S_m. All vector fields on $K^{(n)}$ with this tangent property form a (infinite-dimensional) Lie algebra. The investigation of suitable finite-dimensional compact subalgebras of this Lie algebra may well be expected to yield useful new Poincaré invariant quantum numbers for classifying n-particle states.

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Appendix

Here we shall fix some notations and conventions concerning the Poincaré group that are not explained in the main text. The quantum mechanical Poincaré group $\mathscr{P}$ (without inversions) is the topological space $\mathbb{R}^4 \times \mathrm{SL}(2, \mathbb{C})$ together with the law of multiplication $(a, A)(a', A') = (a + Aa', AA')$, where the action of the matrix group $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{R}^4$ is given by

$$(Ax)^\mu \equiv \tfrac{1}{2}\mathrm{Tr}\, \sigma^\mu A(\sigma^0 x^0 + \boldsymbol{x} \cdot \boldsymbol{\sigma})A^*$$

for all $x \equiv (x^0, x^1, x^2, x^3) \equiv (x^0, \boldsymbol{x}) \in \mathbb{R}^4$, $\mu \in \{0, 1, 2, 3\}$, where σ^0 is the unit element in $\mathrm{SL}(2, \mathbb{C})$ and $\boldsymbol{\sigma} \equiv (\sigma^1, \sigma^2, \sigma^3)$ is the tripel of Pauli matrices. For any $p \in \mathbb{R}^4$ with $p^0 > 0$ and $p \cdot p \equiv (p^0)^2 - \boldsymbol{p}^2 > 0$ we define the positive Hermitian matrix $A(p) \in \mathrm{SL}(2, \mathbb{C})$ by

$$A(p) \equiv [(m + p^0)\sigma^0 + \boldsymbol{p} \cdot \boldsymbol{\sigma}][2m(m + p^0)]^{-1/2} = \mathrm{e}^{\chi \boldsymbol{n} \cdot \boldsymbol{\sigma}/2},$$

where $m \equiv (p \cdot p)^{1/2}$, $\boldsymbol{n} \equiv \boldsymbol{p}/|\boldsymbol{p}|$, $\tanh \chi = |\boldsymbol{p}|/p^0$. We have $A(p)^{-1}p = (m, \boldsymbol{0})$. For any p of this kind and any $A \in \mathrm{SL}(2, \mathbb{C})$, the Wigner rotation $\mathrm{R}(p, A) \in \mathrm{SU}(2)$ is defined by

$$\mathrm{R}(p, A) \equiv A(p)^{-1} A A(A^{-1}p).$$

For $A \in \mathrm{SL}(2, \mathbb{C})$ and $B \in \mathrm{SU}(2)$ we have $\mathrm{R}(p, AB) = \mathrm{R}(p, A)B$, particularly $\mathrm{R}(p, B) = B$. For the Lorentz transformation $A(p)$ we have

$$\mathrm{R}(q, A(p)) = [1 + v^0 + w^0 + v \cdot w - \mathrm{i}(\boldsymbol{v} \times \boldsymbol{w}) \cdot \boldsymbol{\sigma}][2(1 + v^0)(1 + w^0)(1 + v \cdot w)]^{-1/2}$$

with $v \equiv q(q \cdot q)^{-1/2}$, $w \equiv p(p \cdot p)^{-1/2}$.

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